Black Hole Astrophysics
Chapters 9.2.2&9.4

All figures extracted from online sources of from the textbook.

Laws of Conservation of Charge and Current (Ch9.2.2)

Conservation of Charge

Very much mass density, the charge density is the sum of all species.

$$\rho_q \equiv \sum_i n_i q_i$$

However, in this case, it can be zero everywhere in the plasma.

From the continuity equation in classical physics $\frac{\partial \rho_q}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, we can very easily translate that to $\nabla_\alpha J^\alpha = 0$.

This equation enables us to determine only ρ_q, therefore we will later introduce the Generalized Ohm's law to find the $\vec{J}$ part.

*Note：
I sort of find it confusing to have all these 3-vectors and 4-vectos together in 1 slide, so to avoid having to label which is a 3-vector and which is a 4-vector, I chose to put vector/tensor symbols only for 3-vectors and all 4-vectors will be written in component form.

Conservation of Charge

Without loss of generality, we can separate the 4-current into

$$J^{\alpha} = \rho_q U^{\alpha} + \mathfrak{J}^{\alpha}$$

Containing a component parallel to U and another perpendicular to it.

Therefore, $\mathfrak{J}^{\alpha} \equiv P^{\alpha\beta} J_{\beta}$

And it automatically satisfies $U^{\alpha}\mathfrak{J}_{\alpha} = 0$ as we derived last week for the projection tensor.

Recap for Plasma Astrophysics

Recall that we had defined the velocity of a fluid element as $\vec{v} = \frac{m_e n_e \vec{v}_e + m_i n_i \vec{v}_i}{\rho_m} \approx \vec{v}_i$, this is due to the fact that momentum is dominated by protons and therefore the center of momentum velocity can be approximated by the proton velocity.

Next, the current is defined as $\vec{J} = q_e n_e \vec{v}_e + q_i n_i \vec{v}_i \approx q_e n_e (\vec{v}_e - \vec{v}_i)$. The interesting thing to observe here is that given not too much of a density difference between the protons and electrons, the current becomes proportional to the difference of velocities $(\vec{v}_e - \vec{v}_i)$.

A useful (at least I find it useful…) way of thinking of this is that current, $\vec{J}$ is caused motion of electrons with respect to the c.m. velocity (which is $\vec{v}_i$) of the fluid elements.

Thus,

fluid element motion is driven by protons
current is mainly caused by electrons.

Using these two simple concepts we can understand the Generalized Ohm's law quite simply.

The generalized Ohm's law – Review

The generalized Ohm's law reads as :

$$\frac{\partial \vec{J}}{\partial t} = \boxed{-\frac{q_e}{m_e} \vec{\nabla} \cdot \overleftrightarrow{P}_e} + \frac{n_e {q_e}^2}{m_e} (\vec{E} + \vec{v} \times \vec{B}) + \frac{q_e}{m_e} (\vec{J} \times \vec{B}) - \frac{n_e {q_e}^2}{m_e} \hat{\eta} \cdot \vec{J}$$

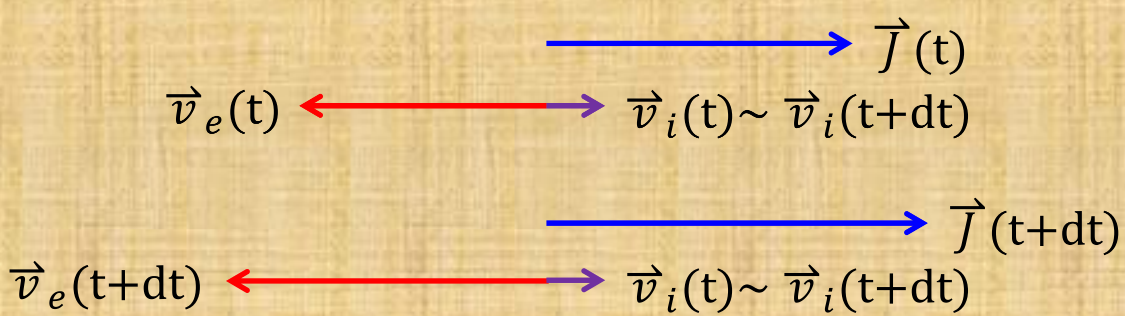
In simple language, it simply says:

What are the different ways we can cause the current to change? $\left(\frac{\partial \vec{J}}{\partial t}\right) = \cdots$

1. The Pressure gradient term $\left(-\frac{q_e}{m_e} \vec{\nabla} \cdot \overleftrightarrow{P}_e\right)$

As we just explained, current is caused by velocity difference between e^-&p^+, however, if the pressure on e^-&p^+ are similar, p^+, being heavier, is harder to push.

Thus, mostly it's only the electrons being accelerated.



$q_e n_e$ omitted for all figures

The generalized Ohm's law – Review

The generalized Ohm's law reads as :

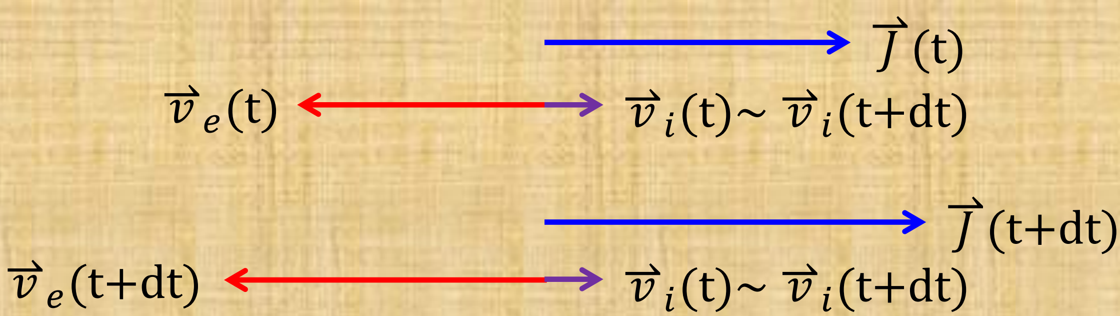
$$\frac{\partial \vec{J}}{\partial t} = -\frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e + \frac{n_e {q_e}^2}{m_e}(\vec{E} + \vec{v}\times\vec{B}) + \frac{q_e}{m_e}(\vec{J}\times\vec{B}) \boxed{-\frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}}$$

In simple language, it simply says:

What are the different ways we can cause the current to change? $\left(\frac{\partial \vec{J}}{\partial t}\right) = \cdots$

2. The collision term $\left(-\frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}\right)$

Again, similar to the pressure gradient term, forces mainly affect the velocity of electrons.



The generalized Ohm's law – Review

The generalized Ohm's law reads as :

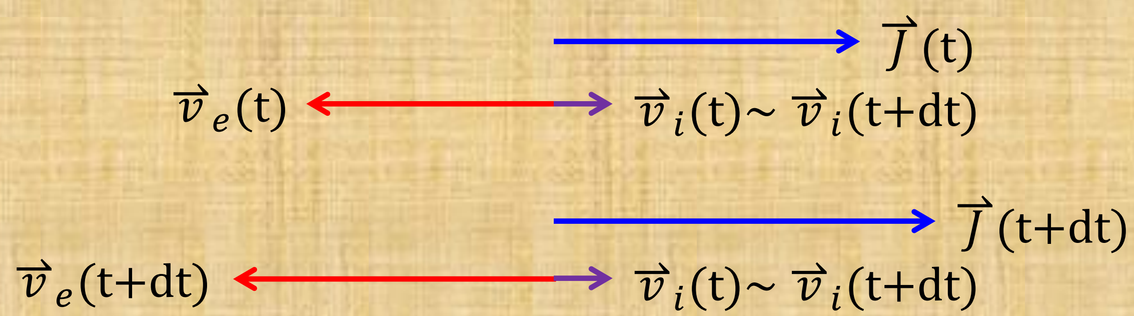
$$\frac{\partial \vec{J}}{\partial t} = -\frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e + \boxed{\frac{n_e {q_e}^2}{m_e}(\vec{E}} + \vec{v}\times\vec{B}) + \frac{q_e}{m_e}(\vec{J}\times\vec{B}) - \frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

In simple language, it simply says:

What are the different ways we can cause the current to change? $\left(\frac{\partial \vec{J}}{\partial t}\right) = \cdots$

3. The Electric field term $\left(\frac{n_e {q_e}^2}{m_e}\vec{E}\right)$

This again is because that a given electric field will mainly accelerate electrons.



The generalized Ohm's law – Review

The generalized Ohm's law reads as :

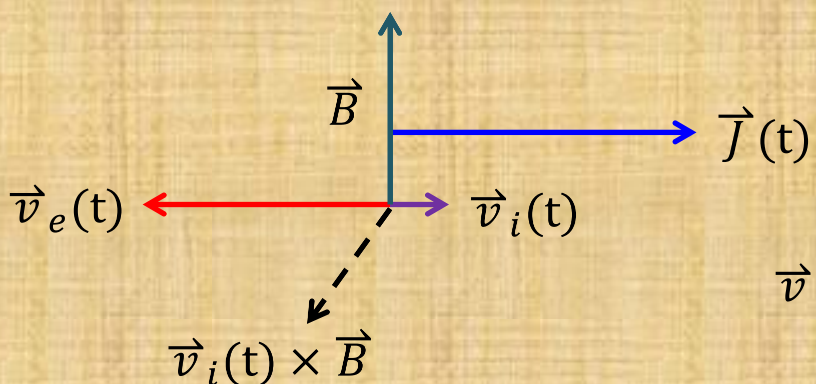
$$\frac{\partial \vec{J}}{\partial t} = -\frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e + \frac{n_e {q_e}^2}{m_e}\left(\vec{E} + \boxed{\vec{v}\times\vec{B}}\right) + \frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right) - \frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

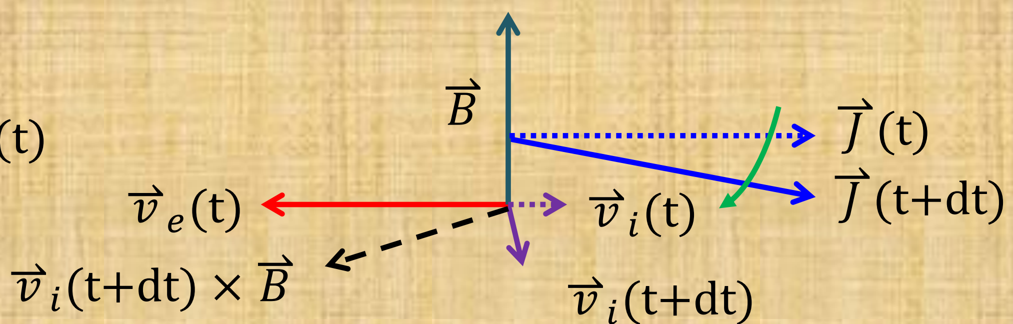
In simple language, it simply says:

What are the different ways we can cause the current to change? $\left(\frac{\partial \vec{J}}{\partial t}\right) = \cdots$

4. The plasma motion term $\left(\frac{n_e {q_e}^2}{m_e}\vec{v}\times\vec{B}\right)$

As we mentioned earlier, the plasma bulk velocity mainly follows that of protons, thus this term tells us the change in current due to protons deflected by B field.





The generalized Ohm's law – Review

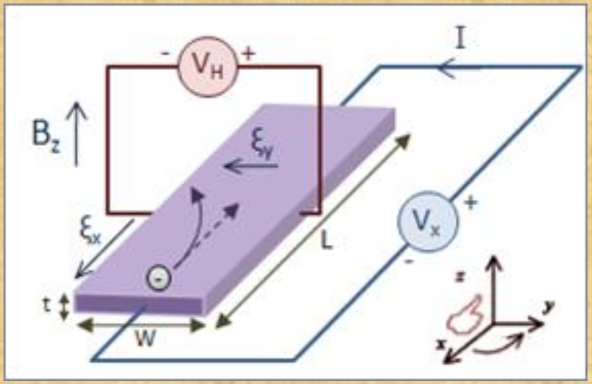
The generalized Ohm's law reads as :

$$\frac{\partial \vec{J}}{\partial t} = -\frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e + \frac{n_e {q_e}^2}{m_e}\left(\vec{E} + \vec{v}\times\vec{B}\right) + \boxed{\frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right)} - \frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

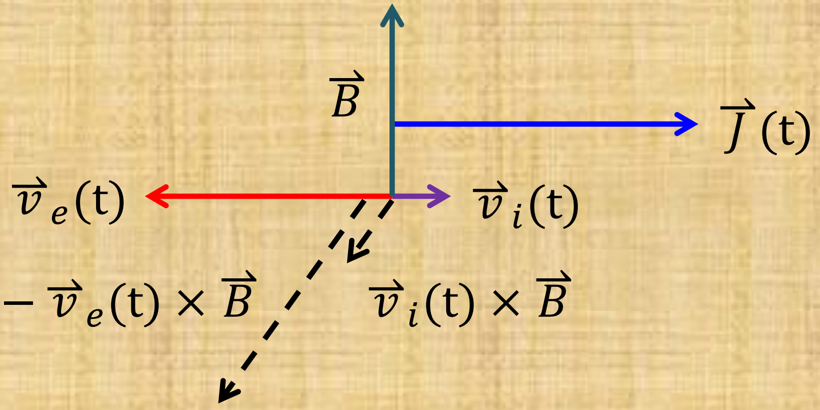
In simple language, it simply says:

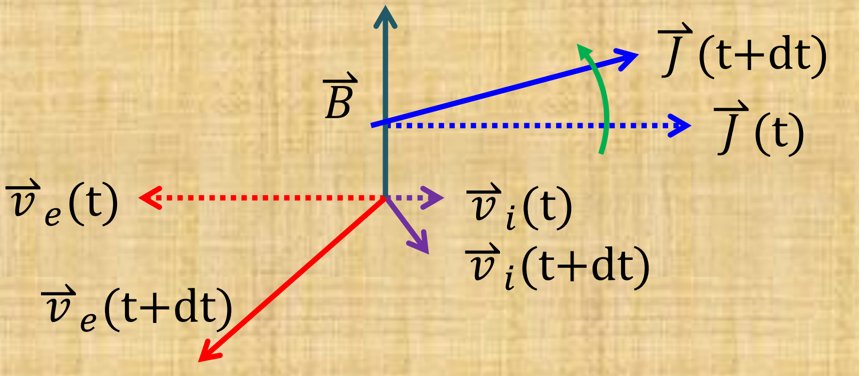
What are the different ways we can cause the current to change? $\left(\frac{\partial \vec{J}}{\partial t}\right) = \cdots$

5. The Hall effect term $\left(\frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right)\right)$



This term is slightly more complicated, it contains $\vec{J}$, which means that both rotation of electron velocity and ion velocity contribute. (It is that Hall effect which is commonly used to determine semiconductor type)





The generalized Ohm's law

Slightly modified, the generalized Ohm's law reads as :

$$\frac{\partial \vec{J}}{\partial t}+\frac{q_e}{m_e}\,\vec{\nabla}\cdot\overleftrightarrow{P}_e=\frac{n_e {q_e}^2}{m_e}\left(\vec{E}+\vec{v}\times\vec{B}\right)+\frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right)-\frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

Now, the fully General Relativistic version:

$$\nabla_\alpha C^{\alpha\beta}=\frac{\omega_p^2}{4\pi}\left(\frac{1}{c}\left(U_\alpha+h_q\mathfrak{J}_\alpha\right)F^{\alpha\beta}-\eta_q\left(\rho_q U^\beta+\mathfrak{J}^\beta\right)\right)$$

$$C^{\alpha\beta}\equiv\left(\rho_q+\frac{\varepsilon_q+p_q}{c^2}\right)U^\alpha U^\beta+U^\alpha(\mathfrak{J}')^\beta+(\mathfrak{J}')^\alpha U^\beta+p_q g^{\alpha\beta}$$

In the rest frame of the fluid, it reads as

$$C^{\alpha\beta}=\begin{pmatrix}\rho_q+\frac{\varepsilon_q}{c^2} & (\mathfrak{J}')^x & (\mathfrak{J}')^y & (\mathfrak{J}')^z\\ (\mathfrak{J}')^x & p_q & 0 & 0\\ (\mathfrak{J}')^y & 0 & p_q & 0\\ (\mathfrak{J}')^z & 0 & 0 & p_q\end{pmatrix}$$

OK… I have to admit, my first response to this was … WTH

The generalized Ohm's law

$$\nabla_\alpha C^{\alpha\beta} = \frac{\omega_p^2}{4\pi}\left(\boxed{\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta}} - \boxed{\eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)}\right)$$

$$C^{\alpha\beta} \equiv \left(\rho_q + \frac{\varepsilon_q + p_q}{c^2}\right)U^\alpha U^\beta + U^\alpha (\mathfrak{J}')^\beta + (\mathfrak{J}')^\alpha U^\beta + p_q g^{\alpha\beta}$$

$$\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} = \left(1, -h_q\frac{\mathfrak{J}_x}{c}, -h_q\frac{\mathfrak{J}_y}{c}, -h_q\frac{\mathfrak{J}_z}{c}\right)\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} \frac{h_q}{c}\left(\vec{\mathfrak{J}}\cdot\vec{E}\right) \\ E_x + \frac{h_q}{c}\left(\mathfrak{J}_y B_z - \mathfrak{J}_z B_y\right) \\ E_y + \frac{h_q}{c}\left(\mathfrak{J}_z B_x - \mathfrak{J}_x B_z\right) \\ E_z + \frac{h_q}{c}\left(\mathfrak{J}_x B_y - \mathfrak{J}_y B_x\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{h_q}{c}\left(\vec{\mathfrak{J}}\cdot\vec{E}\right) \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^x \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^y \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^z \end{pmatrix}$$

$$\eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right) = \eta_q \begin{pmatrix} \rho_q c \\ \mathfrak{J}_x \\ \mathfrak{J}_y \\ \mathfrak{J}_z \end{pmatrix}$$

The generalized Ohm's law

$$\boxed{\nabla_\alpha C^{\alpha\beta}} = \frac{\omega_p^2}{4\pi}\left(\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} - \eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)\right)$$

$$C^{\alpha\beta} \equiv \left(\rho_q + \frac{\varepsilon_q + p_q}{c^2}\right)U^\alpha U^\beta + U^\alpha (\mathfrak{J}')^\beta + (\mathfrak{J}')^\alpha U^\beta + p_q g^{\alpha\beta}$$

$$\nabla_\alpha C^{\alpha\beta} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} \rho_q + \frac{\varepsilon_q}{c^2} & (\mathfrak{J}')^x & (\mathfrak{J}')^y & (\mathfrak{J}')^z \\ (\mathfrak{J}')^x & p_q & 0 & 0 \\ (\mathfrak{J}')^y & 0 & p_q & 0 \\ (\mathfrak{J}')^z & 0 & 0 & p_q \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\varepsilon_q}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{J}}\,' \\ \frac{\partial (\mathfrak{J}')^x}{\partial t} + \frac{\partial p_q}{\partial x} \\ \frac{\partial (\mathfrak{J}')^y}{\partial t} + \frac{\partial p_q}{\partial y} \\ \frac{\partial (\mathfrak{J}')^z}{\partial t} + \frac{\partial p_q}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\varepsilon_q}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{J}}\,' \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^x \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^y \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^z \end{pmatrix}$$

The generalized Ohm's law—putting it together

$$\nabla_\alpha C^{\alpha\beta} = \frac{\omega_p^2}{4\pi}\left(\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} - \eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)\right)$$

$$C^{\alpha\beta} \equiv \left(\rho_q + \frac{\varepsilon_q + p_q}{c^2}\right)U^\alpha U^\beta + U^\alpha(\mathfrak{J}')^\beta + (\mathfrak{J}')^\alpha U^\beta + p_q g^{\alpha\beta}$$

I have no idea what these two terms mean…

The GR version of the generalized Ohm's law in the "Local frame of the fluid"

$$\begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\varepsilon_q}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{J}}\,' \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^x \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^y \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla p_g\right)^z \end{pmatrix} = \frac{\omega_p^2}{4\pi}\begin{pmatrix} \frac{h_q}{c}\left(\vec{\mathfrak{J}}\cdot\vec{E}\right) - \eta_q\rho_q c \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^x - \eta_q\mathfrak{J}^x \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^y - \eta_q\mathfrak{J}^y \\ \left(\vec{E} + \frac{h_q}{c}\left(\vec{\mathfrak{J}}\times\vec{B}\right)\right)^z - \eta_q\mathfrak{J}^z \end{pmatrix}$$

In the local frame, $\vec{v} = 0$

The generalized Ohm's law in classical theory.

$$\frac{\partial \vec{J}}{\partial t} + \frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e = \frac{n_e {q_e}^2}{m_e}\left(\vec{E} + \vec{v}\times\vec{B}\right) + \frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right) - \frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

Pressure Gradient

Plasma motion

Hall

Resistive

Comparison with the 3D version

$$\frac{\partial \vec{J}}{\partial t} + \frac{q_e}{m_e}\vec{\nabla}\cdot\overleftrightarrow{P}_e = \frac{n_e {q_e}^2}{m_e}\left(\underline{\vec{E}+\vec{v}\times\vec{B}}\right) + \frac{q_e}{m_e}\left(\vec{J}\times\vec{B}\right) - \frac{n_e {q_e}^2}{m_e}\hat{\eta}\cdot\vec{J}$$

One final comment on charge dynamics is useful, before we move on to simpler versions of equation (9.33). The *non*-relativistic version of that equation *is* well known and is often referred to as the "generalized Ohm's law" (see, *e.g.*, [350])

Convective terms that were ignored in Plasma class.

$$\frac{\partial \boldsymbol{J}}{\partial t} + \underline{\nabla\cdot\left(\boldsymbol{V}\,\boldsymbol{J} + \boldsymbol{J}\,\boldsymbol{V} - \rho_q\,\boldsymbol{V}\,\boldsymbol{V}\right)} + \nabla p_q =$$

$$\frac{\omega_p^2}{4\pi}\left(\underline{\boldsymbol{E} + \frac{\boldsymbol{V}}{c}\times\boldsymbol{B}} + h_q\,\frac{\boldsymbol{J}}{c}\times\boldsymbol{B} - \eta_q\boldsymbol{J}\right)$$

In a general frame, the $\vec{v}\times\vec{B}$ term comes back.

This is the non-relativistic limit of equation (9.33) and an be obtained by assuming a flat metric ($\boldsymbol{g} = \boldsymbol{\eta}$), slow bulk speed ($|\boldsymbol{V}| \ll c$), and slow particle thermal and drift velocities ($(\varepsilon_q + p_q) \ll \rho_q c^2$ and $\gamma_q = 1$). Note that $\boldsymbol{J}$ is a three-vector, representing the three spatial components of $\mathfrak{J}$.

The terms in the Generalized Ohm's Law

The GR version of the generalized Ohm's law in the "Local frame of the fluid"

$$\begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\varepsilon_q}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{J}}\,' \\ \left(\frac{\partial \boxed{\mathfrak{J}'}}{\partial t} + \nabla p_g\right)^x \\ \left(\frac{\partial \boxed{\mathfrak{J}'}}{\partial t} + \nabla p_g\right)^y \\ \left(\frac{\partial \boxed{\mathfrak{J}'}}{\partial t} + \nabla p_g\right)^z \end{pmatrix} = \frac{\omega_p^2}{4\pi} \begin{pmatrix} \frac{h_q}{c}\left(\boxed{\vec{\mathfrak{J}}}\cdot\vec{E}\right) - \eta_q \rho_q c \\ \left(\vec{E} + \frac{h_q}{c}\left(\boxed{\vec{\mathfrak{J}}}\times\vec{B}\right)\right)^x - \eta_q \mathfrak{J}^x \\ \left(\vec{E} + \frac{h_q}{c}\left(\boxed{\vec{\mathfrak{J}}}\times\vec{B}\right)\right)^y - \eta_q \mathfrak{J}^y \\ \left(\vec{E} + \frac{h_q}{c}\left(\boxed{\vec{\mathfrak{J}}}\times\vec{B}\right)\right)^z - \eta_q \mathfrak{J}^z \end{pmatrix}$$

Spatial current $\vec{\mathfrak{J}} \equiv \sum_i q_i \int \vec{v}\, f(\vec{v}) d^3\, \vec{v}$

Lorentz enhanced version of the spatial current $\vec{\mathfrak{J}}\,' \equiv \sum_i q_i \int \gamma\, \vec{v}\, f(\vec{v}) d^3\, \vec{v}$

In order to "close" the equations (*i.e.*, not end up with more unknowns than equations and have to generate even higher-order moments of the relativistic Boltzmann equation), we will make the approximation that $\mathfrak{J}$ and $\mathfrak{J}'$ are parallel and differ only by a single average Lorentz current beaming factor

$$\boxed{\mathfrak{J}' = \gamma_q \mathfrak{J}} \tag{9.35}$$

The terms in the Generalized Ohm’s Law

The GR version of the generalized Ohm’s law in the “Local frame of the fluid”

$$\begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\boxed{\varepsilon_q}}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{J}}\,' \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla \boxed{p_g}\right)^x \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla \boxed{p_g}\right)^y \\ \left(\frac{\partial \mathfrak{J}'}{\partial t} + \nabla \boxed{p_g}\right)^z \end{pmatrix} = \frac{\boxed{\omega_p^2}}{4\pi} \begin{pmatrix} \frac{\boxed{h_q}}{c}(\vec{\mathfrak{J}}\cdot\vec{E}) - \boxed{\eta_q}\rho_q c \\ \left(\vec{E} + \frac{\boxed{h_q}}{c}(\vec{\mathfrak{J}}\times\vec{B})\right)^x - \boxed{\eta_q}\mathfrak{J}^x \\ \left(\vec{E} + \frac{\boxed{h_q}}{c}(\vec{\mathfrak{J}}\times\vec{B})\right)^y - \boxed{\eta_q}\mathfrak{J}^y \\ \left(\vec{E} + \frac{\boxed{h_q}}{c}(\vec{\mathfrak{J}}\times\vec{B})\right)^z - \boxed{\eta_q}\mathfrak{J}^z \end{pmatrix}$$

Plasma frequency $\qquad \omega_p^2 \equiv 4\pi \sum_i \frac{{q_i}^2 n_i}{m_i} \approx 4\pi \frac{n_e e^2}{m_e}$

Hall coefficient $\qquad h_q \equiv \frac{4\pi}{\omega_p^2 |\vec{\mathfrak{J}}|} \sum_i \frac{q_i}{m_i} |\vec{\mathfrak{J}}_i| \approx \frac{1}{n_e e}$

Resistivity $\qquad \eta_q \equiv 4\pi \frac{\nu_{\mathrm{coll}}}{\omega_p^2}$

These terms can be determined through the equation of states:

$$\varepsilon_q = \varepsilon_q(\rho, T) \qquad p_q = p_q(\rho, T)$$
$$\omega_p = \omega_p(\rho, T) \qquad h_q = h_q(\rho, T)$$
$$\eta_q = \eta_q(\rho, T)$$

Full stress-energy tensor for gas

$$T^{\alpha\beta}{}_{\text{gas}} = T^{\alpha\beta}{}_{\text{fluid}} + T^{\alpha\beta}{}_{\text{Conduction}} + T^{\alpha\beta}{}_{\text{Viscosity}}$$

$$= \left(\left(\rho + \mathrm{p} + \frac{\varepsilon}{c^2}\right) U^\alpha U^\beta + g^{\alpha\beta} p\right) + \left(\frac{1}{c^2}\left(Q_g^\alpha U^\beta + U^\alpha Q_g^\beta\right)\right) + \left(-2\eta_{v,g}\Sigma^{\alpha\beta} - \zeta_{v,g}\Theta P^{\alpha\beta}\right)$$

$$\left(T^{\alpha\beta}\right)_{\text{fluid}} = \begin{pmatrix} \rho + \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$\left(T^{\alpha\beta}\right)_{\text{Conduction}} = \begin{pmatrix} 0 & Q_g^x & Q_g^y & Q_g^z \\ Q_g^x & 0 & 0 & 0 \\ Q_g^y & 0 & 0 & 0 \\ Q_g^z & 0 & 0 & 0 \end{pmatrix}$$

$$\left(T^{\alpha\beta}\right)_{\text{Viscosity}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\eta_{v,g}\Sigma^{\mathrm{xx}} - \zeta_{v,g}\Theta & -2\eta_{v,g}\Sigma^{\mathrm{xy}} & -2\eta_{v,g}\Sigma^{\mathrm{xz}} \\ 0 & -2\eta_{v,g}\Sigma^{\mathrm{yx}} & -2\eta_{v,g}\Sigma^{\mathrm{yy}} - \zeta_{v,g}\Theta & -2\eta_{v,g}\Sigma^{\mathrm{yz}} \\ 0 & -2\eta_{v,g}\Sigma^{\mathrm{zx}} & -2\eta_{v,g}\Sigma^{\mathrm{zy}} & -2\eta_{v,g}\Sigma^{\mathrm{zz}} - \zeta_{v,g}\Theta \end{pmatrix}$$

$$\left(T^{\alpha\beta}\right)_{\text{gas}} = \begin{pmatrix} \rho c^2 + \varepsilon_g & Q_g^x & Q_g^y & Q_g^z \\ Q_g^x & -2\eta_{v,g}\Sigma^{\mathrm{xx}} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{\mathrm{xy}} & -2\eta_{v,g}\Sigma^{\mathrm{xz}} \\ Q_g^y & -2\eta_{v,g}\Sigma^{\mathrm{yx}} & -2\eta_{v,g}\Sigma^{\mathrm{yy}} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{\mathrm{yz}} \\ Q_g^z & -2\eta_{v,g}\Sigma^{\mathrm{zx}} & -2\eta_{v,g}\Sigma^{\mathrm{zy}} & -2\eta_{v,g}\Sigma^{\mathrm{zz}} - \zeta_{v,g}\Theta + p_g \end{pmatrix}$$

Structure similarity with the energy-momentum tensor

$$\nabla_\alpha C^{\alpha\beta} = \frac{\omega_p^2}{4\pi}\left(\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} - \eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)\right)$$

$$C^{\alpha\beta} \equiv \left(\rho_q + \frac{\varepsilon_q + p_q}{c^2}\right)U^\alpha U^\beta + U^\alpha (\mathfrak{J}')^\beta + (\mathfrak{J}')^\alpha U^\beta + p_q g^{\alpha\beta}$$

In the rest frame of the fluid, $C^{\alpha\beta} = \begin{pmatrix} \rho_q + \frac{\varepsilon_q}{c^2} & (\mathfrak{J}')^x & (\mathfrak{J}')^y & (\mathfrak{J}')^z \\ (\mathfrak{J}')^x & p_q & 0 & 0 \\ (\mathfrak{J}')^y & 0 & p_q & 0 \\ (\mathfrak{J}')^z & 0 & 0 & p_q \end{pmatrix}$

$$\left(T^{\alpha\beta}\right)_{\text{gas}} = \begin{pmatrix} \rho c^2 + \varepsilon_g & Q_g^x & Q_g^y & Q_g^z \\ Q_g^x & -2\eta_{v,g}\Sigma^{xx} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{xy} & -2\eta_{v,g}\Sigma^{xz} \\ Q_g^y & -2\eta_{v,g}\Sigma^{yx} & -2\eta_{v,g}\Sigma^{yy} - \zeta_{v,g}\Theta + p_g & -2\eta_{v,g}\Sigma^{yz} \\ Q_g^z & -2\eta_{v,g}\Sigma^{zx} & -2\eta_{v,g}\Sigma^{zy} & -2\eta_{v,g}\Sigma^{zz} - \zeta_{v,g}\Theta + p_g \end{pmatrix}$$

The conservation of beamed current has all the same physics as the conservation of energy-momentum, *except* for the spatial viscous terms (*i.e.*, $\mathcal{T}_{\text{VISC}}$). This is because the contribution of viscous collisional effects were not considered in the derivation of equation (D.14), not because such viscous effects on charge dynamics do not exist.

How do we solve them?

For simplicity, let’s look at the local frame of the fluid. The generalized Ohm’s law then looks like:

$$\begin{pmatrix} \frac{\partial}{\partial t}\left(\rho_q + \frac{\varepsilon_q}{c^2}\right) + \vec{\nabla}\cdot\vec{\mathfrak{I}}\,' \\ \left(\frac{\partial \mathfrak{I}'}{\partial t} + \nabla p_g\right)^x \\ \left(\frac{\partial \mathfrak{I}'}{\partial t} + \nabla p_g\right)^y \\ \left(\frac{\partial \mathfrak{I}'}{\partial t} + \nabla p_g\right)^z \end{pmatrix} = \frac{\omega_p^2}{4\pi} \begin{pmatrix} \frac{h_q}{c}(\vec{\mathfrak{I}}\cdot\vec{E}) - \eta_q\rho_q c \\ \left(\vec{E} + \frac{h_q}{c}(\vec{\mathfrak{I}}\times\vec{B})\right)^x - \eta_q\mathfrak{I}^x \\ \left(\vec{E} + \frac{h_q}{c}(\vec{\mathfrak{I}}\times\vec{B})\right)^y - \eta_q\mathfrak{I}^y \\ \left(\vec{E} + \frac{h_q}{c}(\vec{\mathfrak{I}}\times\vec{B})\right)^z - \eta_q\mathfrak{I}^z \end{pmatrix}$$

These three equations allow us to take a time step for $\mathfrak{I}'$.

This equation allows us to determine γ_q. (How?)

$\vec{\mathfrak{I}}\,' = \gamma_q\,\vec{\mathfrak{I}}$ helps us to find $\vec{\mathfrak{I}}$ and $U^\alpha\mathfrak{I}_\alpha = 0$ helps to find $\mathfrak{I}^\alpha$

Then from charge continuity, $\nabla_\alpha J^\alpha = 0$ and $J^\alpha = \rho_q U^\alpha + \mathfrak{I}^\alpha$ we can update ρ_q.

With the new 4-current, we can update the fields using Maxwell’s equations.

Relativistic Hall MHD

The generalized Ohm's law reads as:

$$\nabla_\alpha C^{\alpha\beta} = \frac{\omega_p^2}{4\pi}\left(\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} - \eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)\right)$$

However, in most cases, $\frac{\omega_p^2}{4\pi}$ is very large that we can forget about the $\nabla_\alpha C^{\alpha\beta}$ term.
This reduces the equation to

$$\frac{1}{c}\left(U_\alpha + h_q \mathfrak{J}_\alpha\right)F^{\alpha\beta} = \eta_q\left(\rho_q U^\beta + \mathfrak{J}^\beta\right)$$

In which we find that the beamed current $(\mathfrak{J}')^\beta$ is gone (it is in the $C^{\alpha\beta}$ term)

Therefore, we no longer need γ_q. This in turn means that the 't' component of the Ohm's law which was used to determine γ_q now is useless.

To simplify, we can project out the spatial current $\mathfrak{J}^\beta$:

$$\frac{U_\alpha}{c}F^{\alpha\beta} + h_q \frac{\mathfrak{J}^\gamma}{c}F^{\alpha\beta}P_{\gamma\alpha} = \eta_q \mathfrak{J}^\beta$$

In non-relativistic case, it's simply $\vec{E} + h_q \frac{\vec{J}}{c} \times \vec{B} = \eta_q \vec{J}$

Relativistic Resistive MHD

Another simplification that is often used it to drop the Hall term in the following equation we got just now:

$$\frac{U_\alpha}{c}F^{\alpha\beta} + \underbrace{h_q\frac{\mathfrak{I}^\gamma}{c}F^{\alpha\beta}P_{\gamma\alpha}}_{\text{Hall term}} = \eta_q\mathfrak{I}^\beta$$

By comparing the Hall and resistive(collision) terms,

$$\frac{h_q|\mathfrak{I}^\alpha||\vec{B}|/c}{\eta_q|\mathfrak{I}^\alpha|} \approx \frac{\mathrm{eB}}{m_e c}\frac{1}{\nu_{\mathrm{coll}}} = \frac{\nu_L}{\nu_{\mathrm{coll}}} << 1$$

we can argue that since many collisions can occur within a Larmor orbit, the $\frac{\vec{J}}{c}\times\vec{B}$ term will not contribute much…????????

Plugging in the numbers gives…

$\nu_{\mathrm{L}} = 1.76\times 10^7\mathrm{B}$ for B~1mG, this is of order 10000….

$\nu_{\mathrm{coll}} = \frac{3.41}{\mathrm{T}^{3/2}}\mathrm{n}$ for T~10^8, n~10^{15} this is of order 1000…

What the hell happened with the ratio!?

Relativistic Ideal MHD

The last simplification to make is to consider dropping off the resistive term.

$$\frac{U_\alpha}{c} F^{\alpha\beta} = \underline{\eta_q \mathfrak{J}^\beta}$$

We estimate the conductivity as follows $\left(V_{\mathrm{th}} \approx \sqrt{\frac{\mathrm{kT}}{m_e}}; r_e = \frac{e^2}{\mathrm{kT}}\right)$:

$$\sigma_q \equiv \frac{1}{\eta_q} = \frac{\omega_p^2}{4\pi\nu_{\mathrm{coll}}} \approx \left(\frac{n_e e^2}{m_e}\right) \frac{1}{n_e(\pi \mathrm{r}_e{}^2)V_{\mathrm{th}}} \approx \frac{(2\mathrm{kT})^{\frac{3}{2}}}{\pi \mathrm{e}^2\sqrt{m_e}} \sim 2.1 \times 10^8 s^{-1} T^{1.5}$$

For $T = 10^6 \sim 10^{10} K$, this gives $\sigma_q = 10^{17} \sim 10^{23}\ s^{-1} >> 5 \times 10^{16}$ for copper!

Given the extreme lack of electrical resistance in astrophysical plasmas, one can safely ignore the resistivity term, leading to the simple "ideal" Ohm's law expression

$$\frac{U_\alpha}{c} F^{\alpha\beta} = 0$$

Relativistic Ideal MHD

In the non-relativistic limit and evaluating in the lab frame, $\frac{U_\alpha}{c} F^{\alpha\beta} = 0$ becomes $\vec{E} = -\frac{\vec{v}}{c} \times \vec{B}$ which is the familiar ideal Ohm's law expression.

Ideal MHD has quite a different character, both mathematically and physically, from the other styles of magnetohydrodynamics discussed above. Ohm's law in the ideal case is *not* used to determine the current density $\mathfrak{J}$ directly. Instead, it is used to determine the electric field components in terms of the fluid velocity and *magnetic* field; these then are used in the other conservation laws and Maxwell's equations. The current is determined, then, from the curl of the electric field in Faraday's law.

I personally think that it should not only be Ideal MHD that deserves this comment, starting from the step when we dropped off the $\frac{\partial \vec{J}}{\partial t}$ term to get the Hall-MHD, it becomes basically impossible to directly update the current…

Flux freezing in actual simulations

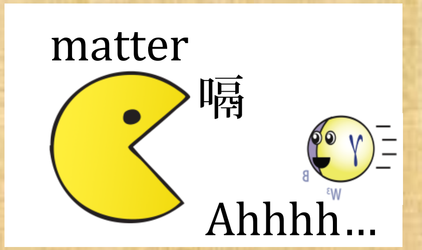
Physically the ideal MHD assumption freezes the magnetic field into the fluid flow, forcing the former to advect with the fluid. Ideal MHD never displays magnetic reconnection and field restructuring. However, because *numerical* simulations always have some artificial numerical resistivity, due to the finite size of the grid spacing, they never can be truly ideal. In fact, usually this numerical resistivity is much greater than the true physical resistivity of plasmas in black hole accretion flows. So the artificial resistivity dominates the simulation, unless a larger physical η_q is assumed and equation (9.41) is used instead of (9.42).

To date, most astrophysical simulations use some form of the ideal MHD approximation (relativistic or not) and, if resistivity and reconnection are needed, these simulations use the artificial resistivity inherent in the numerical method rather than assuming a specific η_q. However, resistive MHD (using equation (9.41)) is much more common in solar plasma physics studies, but even in that case an anomalously large η_q is assumed rather than the 10^{-17}–10^{-23} s values estimated above. However, those few astrophysical simulations that do use charged particle dynamics do not have to use an anomalous resistivity; simulating particle interactions directly appears to produce the proper physics, including reconnection, that occurs in astrophysical plasmas.

Optically thin Radiative Emission (Ch9.4)

Optically thin radiation heat transfer –an Introduction

Scattering

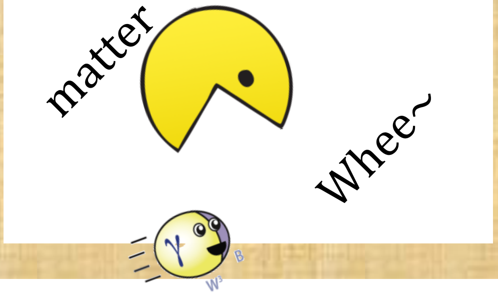


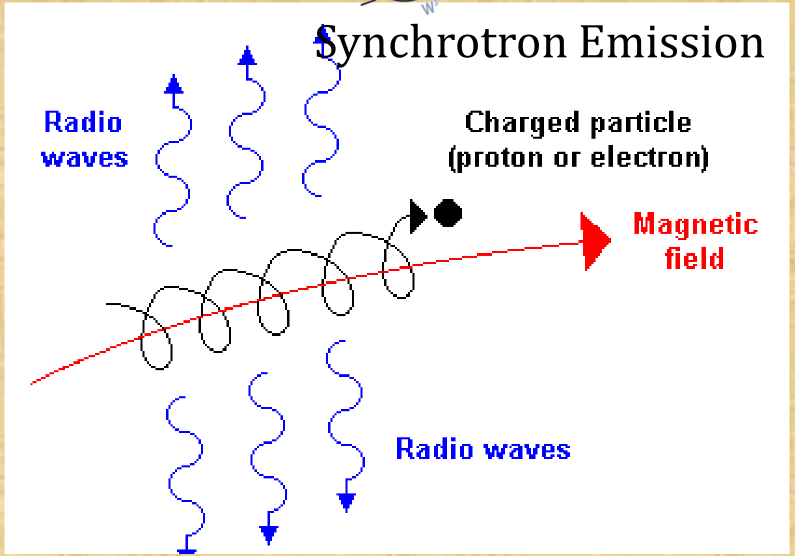
Absroption

Previously, we discuss the heat transfer by radiation by considering the opacity, which means that we're considering radiation transferring heat within the system – being eaten or scattered within the system.

Now we consider photons that are allowed to escape the system and thereby carrying energy with them. And the spectrum remains approximately that of the emission process.

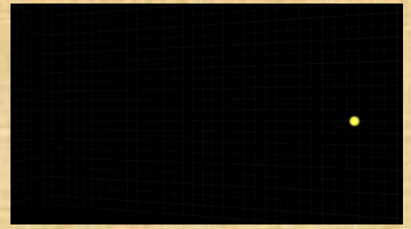
$$\dot{q}_r = \frac{1}{\rho}\vec{\nabla}\cdot\vec{Q}_r = -\frac{c}{3\rho}\vec{\nabla}\cdot\left[\frac{\nabla\varepsilon_r}{\bar{\kappa}_{R,\mathrm{bf}}}\right]$$



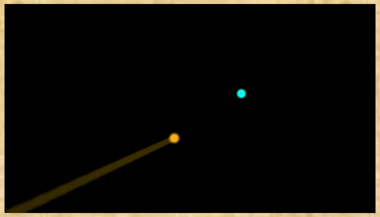


We discuss the following processes:

Inverse Compton

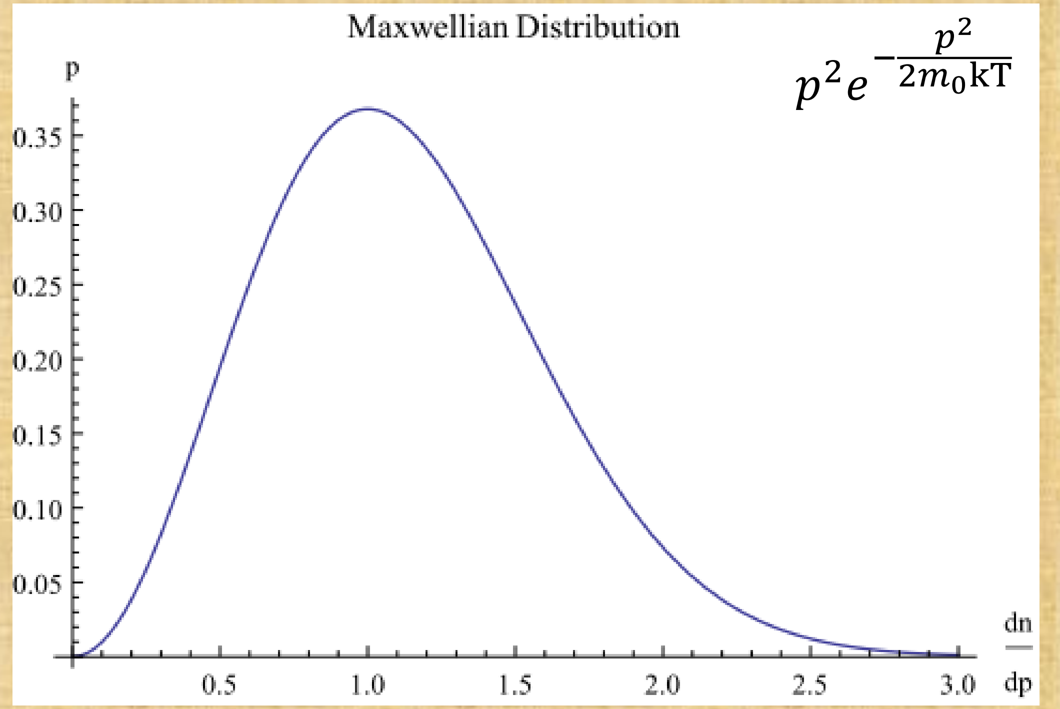


Bremsstrahlung

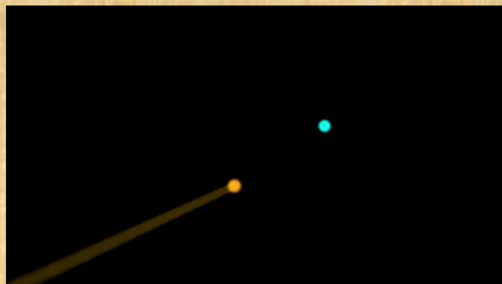


The BIG assumption

We consider the different types of emission radiated from a thermal distribution of particles!
(For example, Maxwellian for a classical gas)



Bremsstrahlung (Free-Free Emission)



$$\dot{q}^-_{\rm br} = \dot{q}^-_{ei} + \dot{q}^-_{ee} + \dot{q}^-_{\pm}$$

$$\dot{q}^-_{ei} = 147\ {\rm erg\ g^{-1}\ s^{-1}}\,(n_{\rm e} + n_+) \times \begin{cases} 1.00\,\theta_{\rm e}^{1/2}(1 + 1.781\,\theta_{\rm e}^{1.34}) & \theta_{\rm e} < 1 \\ 1.41\,\theta_{\rm e}\,[\ln(1.123\,\theta_{\rm e} + 0.48) + 1.5] & \theta_{\rm e} \geq 1 \end{cases}$$

$e^-{-}ion; e^+ -$ ion

$$\dot{q}^-_{ee} = 251\ {\rm erg\ g^{-1}\ s^{-1}}\,\frac{(n_{\rm e}^2 + n_+^2)}{n_{\rm p}} \times \begin{cases} 1.00\,\theta_{\rm e}^{1.5}(1 + 1.1\,\theta_{\rm e} + \theta_{\rm e}^2 - 1.25\,\theta_{\rm e}^{2.5}) & \theta_{\rm e} < 1 \\ 1.33\,\theta_{\rm e}\,[\ln(1.123\,\theta_{\rm e}) + 1.28] & \theta_{\rm e} \geq 1 \end{cases}$$

$e^-e^-; e^+e^+$

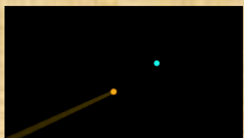
$$\dot{q}^-_{\pm} = 337\ {\rm erg\ g^{-1}\ s^{-1}}\,\frac{n_{\rm e} n_+}{n_{\rm p}} \times \begin{cases} 1.00\,\theta_{\rm e}^{1/2}(1 + 1.7\,\theta_{\rm e}^{3/2}) & \theta_{\rm e} < 1 \\ 1.99\,\theta_{\rm e}\,[\ln(1.123\,\theta_{\rm e}) + 1.24] & \theta_{\rm e} \geq 1 \end{cases}$$

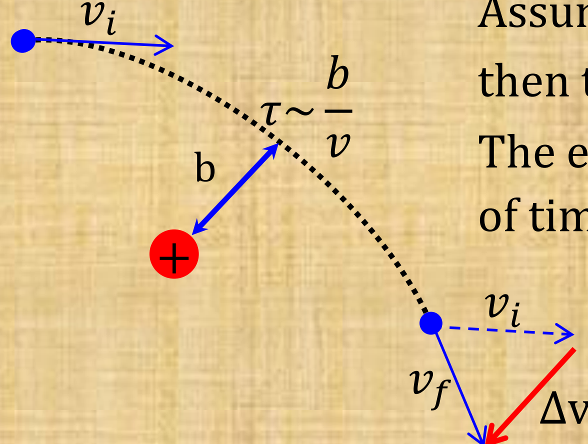
e^-e^+

$\theta_{\rm e} \equiv kT_{\rm e}/m_e c^2$ is the dimensionless electron temperature

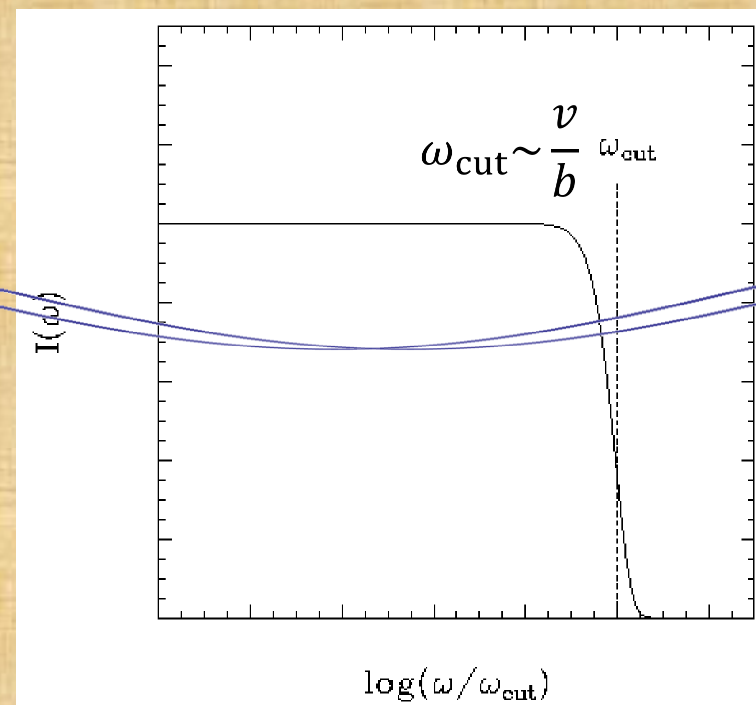
How do we understand these trends in a simpler manner?

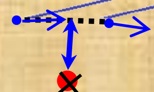
Bremsstrahlung (Free-Free Emission)





Assuming that the v magnitude doesn't change much, then the time that the charge takes to fly past is $\tau \sim \frac{b}{v}$. The emitted frequency (being the Fourier transform of time) therefore would have a cut off at $\omega_{\text{cut}} \sim \frac{v}{b}$.



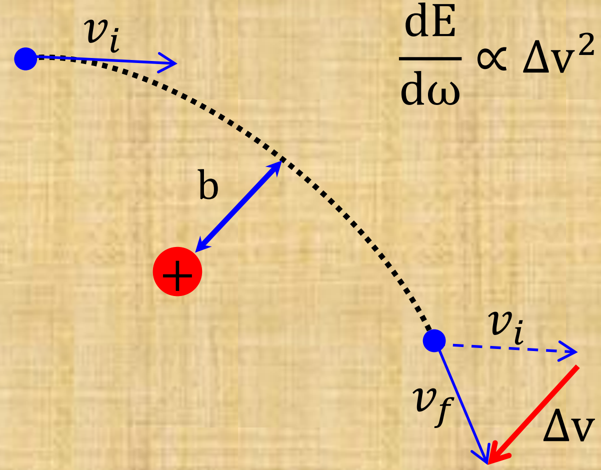


Constructive!

For the low frequency waves, the interaction time is much shorter than the inverse of its frequency. Thus, these waves will be coherent with those emitted a small time ago of the same frequency.

This leads to $\frac{\mathrm{dE}}{\mathrm{d}\omega} \propto \Delta \mathrm{v}^2$

Bremsstrahlung (Free-Free Emission)



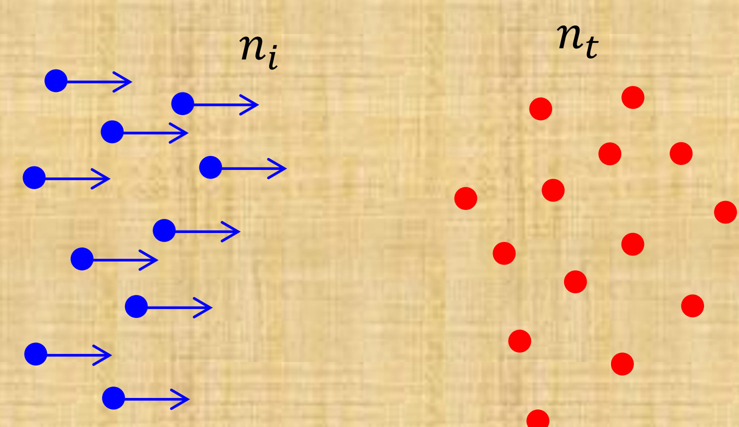
$$\frac{\mathrm{dE}}{\mathrm{d}\omega} \propto \Delta \mathrm{v}^2$$

Now, we can estimate the change in velocity Δv by integrating over the acceleration history of the particle

$$\Delta \mathrm{v} = \frac{\mathrm{Ze}^2}{m}\int_{-\infty}^{\infty} \frac{b\,\mathrm{dt}}{\left(\sqrt{b^2+(\mathrm{vt})^2}\right)^3} = \frac{2Ze^2}{mbv} \propto \frac{1}{mbv}$$

This gives $\displaystyle \frac{\mathrm{dE}}{\mathrm{d}\omega} \propto \frac{1}{m^2 b^2 v^2}$

Since in astrophysics, there are usually a large number of particles, we now consider a flux of particles shooting at stationary targets.



The emitted power per unit volume per unit frequency is then

$$\frac{\mathrm{dE}}{\mathrm{d}\omega \mathrm{dVdt}} \propto \int_{b_{\mathrm{min}}}^{b_{\mathrm{max}}} \frac{1}{m^2 b^2 v^2}\, n_t \underbrace{(n_i v)(2\pi b \mathrm{db})}$$

Number of incident particles per unit time at a particular impact parameter

Thermal Bremsstrahlung

$$\frac{\mathrm{dE}}{\mathrm{d\omega dVdt}} \propto \int_{b_{\min}}^{b_{\max}} \frac{1}{m^2 b^2 v^2}\left[n_t (n_i v)(2\pi b\,\mathrm{db})\right] \propto \frac{n_t n_i}{m^2 v}\,\ell\mathrm{n}\left(\frac{b_{\max}}{b_{\min}}\right)$$

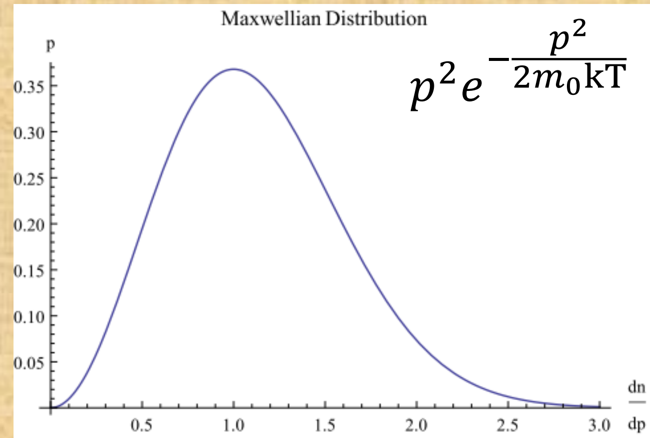
Gaunt factor ($\bar{g}$), in general some complicated function of order unity.

For a thermal population of particles, for a quick estimate, we can characterized the velocity by the thermal velocity

$$v_{\mathrm{th}} \propto \sqrt{\frac{T}{m}}$$

which gives

$$\frac{\mathrm{dE}}{\mathrm{d\omega dVdt}} \propto \frac{n_t n_i}{m^{1.5}\sqrt{T}}\,\bar{g}$$



Thermal Bremsstrahlung—total power

$$\frac{\mathrm{dE}}{\mathrm{dVdt}} = \int \frac{\mathrm{dE}}{\mathrm{d\omega dVdt}} \mathrm{d\omega} \propto \int \frac{n_t n_i}{m^{1.5}\sqrt{T}} \bar{g} \mathrm{d\omega}$$

$$\approx \frac{n_t n_i}{m^{1.5}\sqrt{T}} \bar{g} \omega_{\mathrm{cut}}$$

For ω_{cut}, the 'b' that gives the highest frequency is obviously the smallest b, i.e. b_{min}.

Here, we can have two cases:

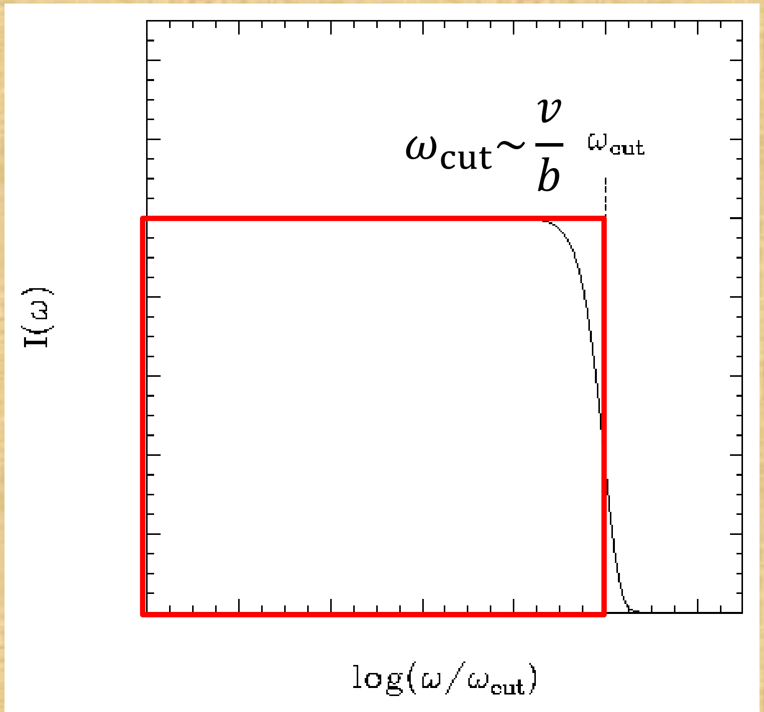
1. The emitting particles have less energy and a large portion is radiated.

$$b_{\mathrm{min}}^{(1)} = \frac{h}{\mathrm{mv}} \rightarrow \frac{v}{b_{\mathrm{min}}^{(1)}} \propto \frac{\mathrm{mv}^2}{h} \propto \frac{T}{h}$$

2. The emitting particles are very energetic and only give away a small portion of their kinetic energy to radiation.

$$b_{\mathrm{min}}^{(2)} \propto \frac{1}{\mathrm{mv}^2} \rightarrow \frac{v}{b_{\mathrm{min}}^{(2)}} \propto \mathrm{mv}^3 \propto \frac{T^{1.5}}{\sqrt{m}}$$

Using the fact that the spectra can be approximated by a box function,



This idea works but seems slightly incompatible to the picture given in Rybicki & Lightman p158…

Thermal Bremsstrahlung—total power

Power per unit volume $\frac{\mathrm{dE}}{\mathrm{dVdt}} \approx \frac{n_t n_i}{m^{1.5}\sqrt{T}} \bar{g}\omega_{\mathrm{cut}}$

1. The emitting particles have less energy and a large portion is radiated.

$$b_{\min}^{(1)} = \frac{h}{\mathrm{mv}} \rightarrow \frac{v}{b_{\min}^{(1)}} \propto \frac{\mathrm{mv}^2}{h} \propto \frac{T}{h}$$

Power per unit volume:

$$\frac{\mathrm{dE}}{\mathrm{dVdt}} \propto \frac{n_t n_i}{h m^{1.5}} \sqrt{T} \bar{g}$$

Power per unit mass:

$$\frac{\mathrm{dE}}{\mathrm{dt\,dm_{tot}}} \propto \frac{1}{m_p n_p} \frac{n_t n_i}{h m^{1.5}} \sqrt{T} \bar{g}$$

2. The emitting particles are very energetic and only give away a small portion of their kinetic energy to radiation.

$$b_{\min}^{(2)} \propto \frac{1}{\mathrm{mv}^2} \rightarrow \frac{v}{b_{\min}^{(2)}} \propto \mathrm{mv}^3 \propto \frac{T^{1.5}}{\sqrt{m}}$$

Power per unit volume:

$$\frac{\mathrm{dE}}{\mathrm{dVdt}} \propto \frac{n_t n_i}{m^2} T \bar{g}$$

Power per unit mass:

$$\frac{\mathrm{dE}}{\mathrm{dt\,dm_{tot}}} \propto \frac{1}{m_p n_p} \frac{n_t n_i}{m^2} \mathrm{T}\, \bar{g}$$

Thermal Bremsstrahlung

Case1: Low energy

$$\frac{dE}{dt\,dm_{tot}} \propto \frac{1}{m_p n_p} \frac{n_t n_i}{h m^{1.5}} \boxed{\sqrt{T}}\, \boxed{\bar{g}}$$

Case2: High energy

$$\frac{dE}{dt\,dm_{tot}} \propto \frac{1}{m_p n_p} \frac{n_t n_i}{m^2} \boxed{T}\, \boxed{\bar{g}}$$

What's caused the difference in this case?

$$\dot{q}_{ei}^{-} = 147 \text{ erg g}^{-1}\text{ s}^{-1} (n_e + n_+) \quad e^- - ion; e^+ - \text{ion}$$

$$\times \begin{cases} 1.00\, \theta_e^{1/2} (1 + 1.781\, \theta_e^{1.34}) & \theta_e < 1 \\ 1.41\, \theta_e [\ln(1.123\, \theta_e + 0.48) + 1.5] & \theta_e \geq 1 \end{cases}$$

$$\dot{q}_{ee}^{-} = 251 \text{ erg g}^{-1}\text{ s}^{-1} \frac{(n_e^2 + n_+^2)}{n_p} \quad e^- e^-; e^+ e^+$$

$$\times \begin{cases} 1.00\, \theta_e^{1.5} (1 + 1.1\, \theta_e + \theta_e^2 - 1.25\, \theta_e^{2.5}) & \theta_e < 1 \\ 1.33\, \theta_e [\ln(1.123\, \theta_e) + 1.28] & \theta_e \geq 1 \end{cases}$$

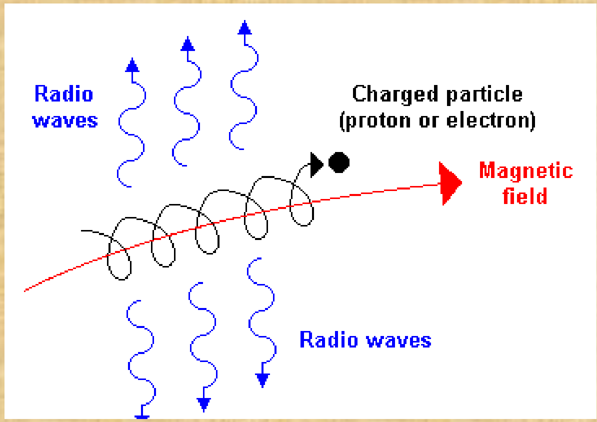
$$\dot{q}_{\pm}^{-} = 337 \text{ erg g}^{-1}\text{ s}^{-1} \frac{n_e n_+}{n_p} \quad e^- e^+$$

$$\times \begin{cases} 1.00\, \theta_e^{1/2} (1 + 1.7\, \theta_e^{3/2}) & \theta_e < 1 \\ 1.99\, \theta_e [\ln(1.123\, \theta_e) + 1.24] & \theta_e \geq 1 \end{cases}$$

$\theta_e \equiv kT_e/m_e c^2$ is the dimensionless electron temperature

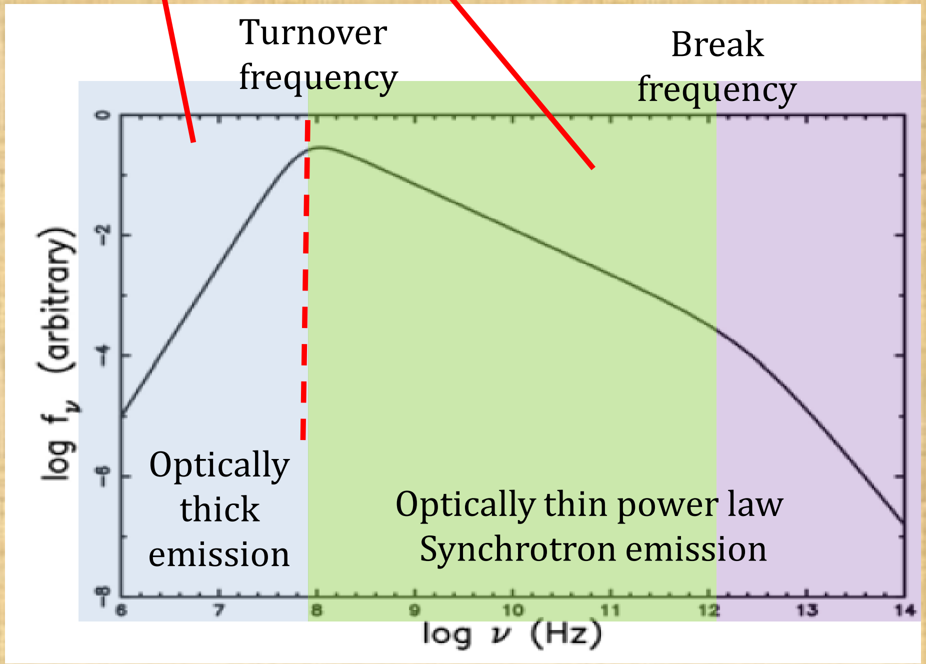
I'm not quite sure whether the $m^{1.5}$ causes the difference between 147 and 337 for the $e^- - ion; e^+ -$ ion v.s. $e^- e^+$ case. Using the reduced mass gives an overestimate of 20%...

Thermal Synchrotron



The local synchrotron cooling rate is approximated as a sum of optically thick and thin emission:

$$\dot{q}_s^- = \frac{2\pi k T_e}{3Hc^2}{\nu_c}^3 + \int_{\nu_c}^{\infty} \epsilon_s(\nu)d\nu$$



*Note that there is some confusion in the units here, $\dot{q}_s^-$ should be erg/g/s but for the synchrotron part, all units are erg/s/cm^3

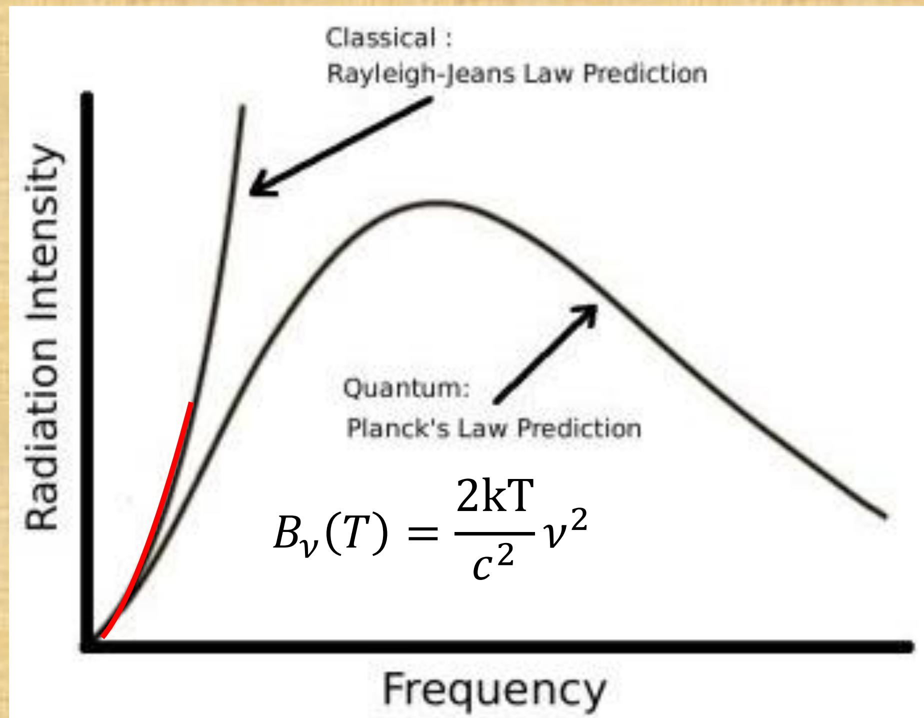
I think just dividing by the mean density of the plasma should fix the small bug:

$$\dot{q}_s^- = \frac{1}{\rho}\left(\frac{2\pi k T_e}{3Hc^2}\nu_c^3 + \int_{\nu_c}^{\infty} \epsilon_s(\nu)\mathrm{d}\nu\right)$$

Therefore, for the Synchrotron section, we will follow the book but keep in mind the small difference.

Thermal Synchrotron –Optically thick part

Since we are considering a Thermal distribution of emitting particles, the optically thick emission can be approximated by Rayleigh-Jeans Law



Then, calculating the power per unit volume gives:

$$\frac{\mathrm{dE}}{\mathrm{dVdt}} \approx \frac{\pi}{H}\int B_\nu(T)\mathrm{d}\nu = \frac{2\pi \mathrm{kT}_e}{3\mathrm{Hc}^2}{\nu_c}^3$$

As given in the textbook.

$$\dot{q}_s^- = \underline{\frac{2\pi \mathrm{kT}_e}{3\mathrm{Hc}^2}{\nu_c}^3} + \int_{\nu_c}^{\infty} \epsilon_s(\nu)d\nu$$

*Note that in most texts, one will see that the intensity of optically thick synchrotron is $I_{\nu,\mathrm{thick}}(\nu) \propto \nu^{2.5}$. That is derived using a Power Law population of emitters, not thermal (as is assumed here)!

Thermal Synchrotron –Optically thin part

and solving the above expression numerically. For an isotropic, Maxwellian distribution of electrons and positrons, the optically thin volume emissivity (for $\nu > \nu_c$) is [354, 353]

$$\boxed{\epsilon_s(\nu, \vartheta) = 4.43 \times 10^{-30}\, 4\pi\nu\, (n_{\rm e} + n_+)\, \frac{I\,(x_M/\sin\vartheta)}{K_2(1/\theta_{\rm e})}\ \mathrm{erg\ cm^{-3}\ s^{-1}}} \qquad (9.89)$$

where $x_M = \nu/\nu_M$ is the normalized frequency (with $\nu_M = 6.27 \times 10^{18}\, B\,(kT_{\rm e})^2$ [cgs] being the critical *electron* frequency for a given temperature), ϑ is the angle between the observer and the magnetic field direction, K_2 is the modified Bessel function of the second kind of order 2, given by the integral

$$K_2(1/\theta_{\rm e}) \equiv \frac{\theta_{\rm e}^2}{3} \int_{1/\theta_{\rm e}}^{\infty} (z^2 - 1/\theta_{\rm e}^2)^{3/2}\, e^{-z}\, dz \qquad (9.90)$$

Normalizing the integration

and the electron-energy-integrated, unitless spectrum is given by [1]

$$I\left(\frac{x_M}{\sin\vartheta}\right) \equiv \frac{\sin\vartheta}{x_M} \int_{1/\theta_{\rm e}}^{\infty} (z^2 - 1/\theta_{\rm e}^2)^{3/2}\, e^{-z}\, \boxed{F(x_M/z^2 \sin\vartheta)}\, dz \qquad (9.91)$$

Integrate over relativistic Maxwellian???

where $F(x)$ is the normalized synchrotron spectrum for a single electron

$$\boxed{F(x) = x \int_x^{\infty} K_{5/3}(\xi)\, d\xi}$$

Radio astrophysics. Nonthermal processes in galactic and extragalactic sources by Pacholczyk, A. G

ApJ...465..327M

1996ApJ...465..312E

Thermal Synchrotron –Optically thin part

In the high-temperature limit ($\theta_e \gg 1$) equation (9.89) can be simplified and made tractable[6]

$$\boxed{\epsilon_s(\nu) = 4.43 \times 10^{-30}\, 2\pi\nu\,(n_e + n_+)\,\frac{I'(x_M)}{\theta_e^2}\ \text{erg cm}^{-3}\ \text{s}^{-1}} \tag{9.92}$$

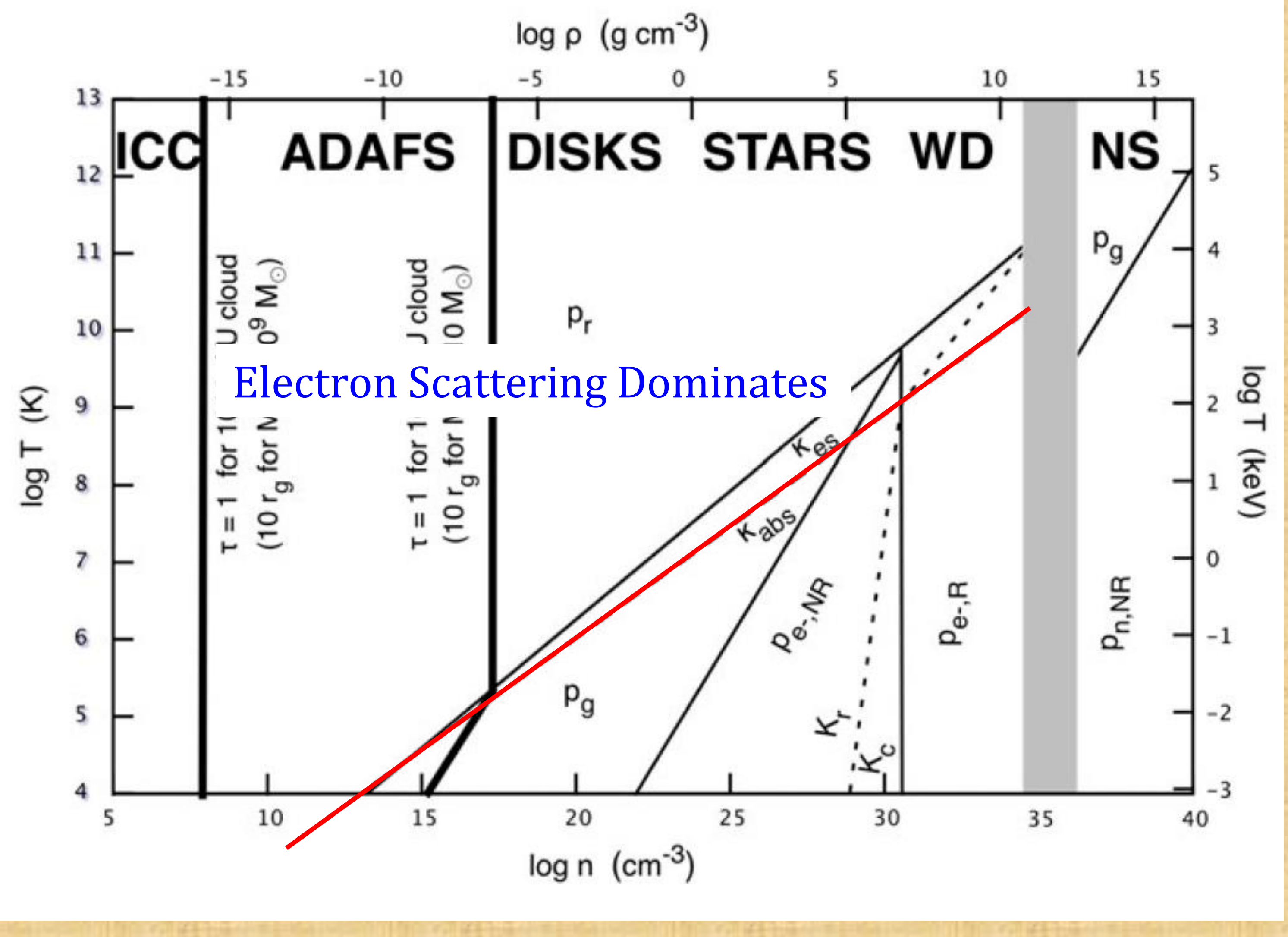
That is, in the limit $1/\theta_e \to 0$, the integral in equation (9.90) becomes $K_2(1/\theta_e) \to 2\,\theta_e^2$, and the integral in equation (9.91) (when also averaged over angle ϑ) can be fit to the following expression [354]

$$I'(x_M) = \frac{4.0505}{x_M^{1/6}}\left(1 + \frac{0.40}{x_M^{1/4}} + \frac{0.5316}{x_M^{1/2}}\right)\exp(-1.8899\,x_M^{1/3}) \tag{9.93}$$

with no more than 2.7% error over the range $0 < x_M < \infty$. Equations (9.92) and (9.93) represent the synchrotron cooling very well over the range in temperature where this type of emission is important, and, *when used together in equation (9.91)*, they avoid a cooling runaway that can produce the wrong physical results in MHD simulations [355].

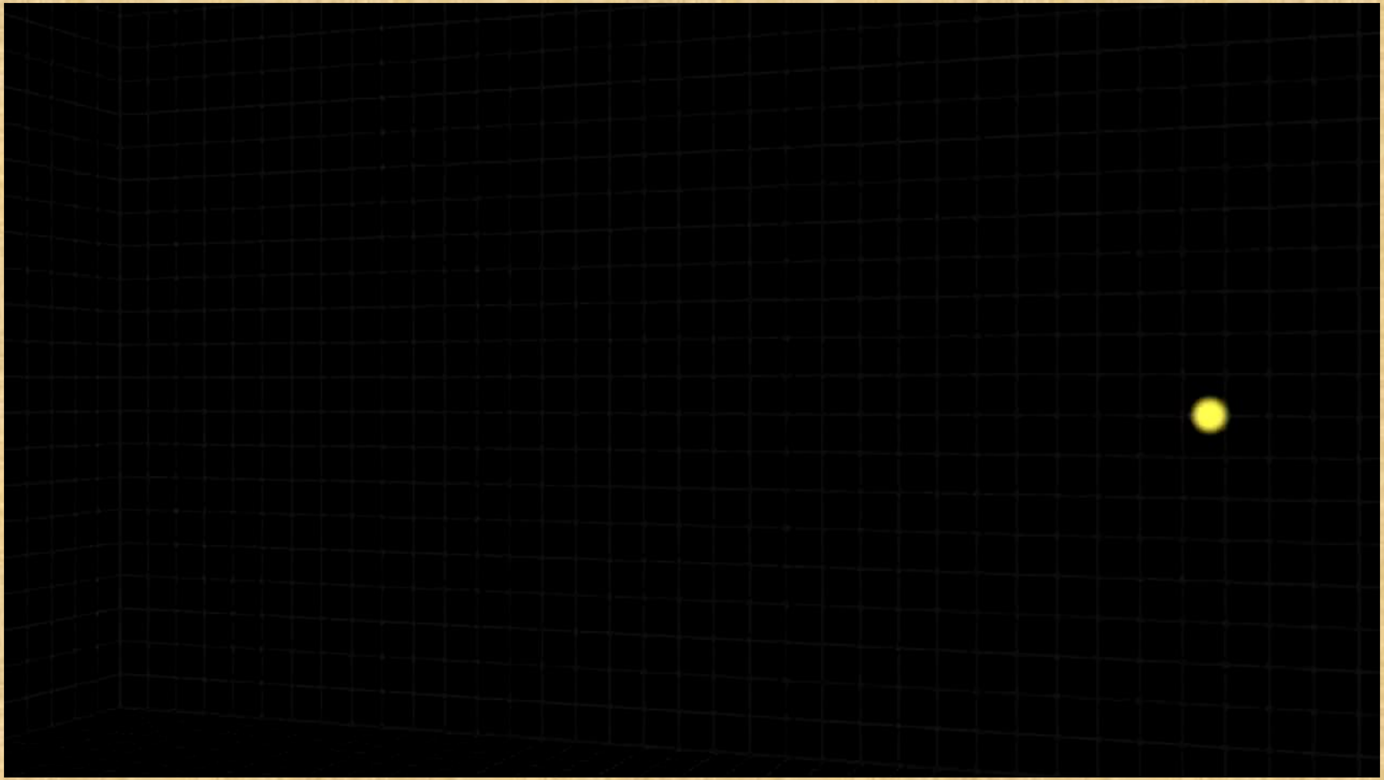
Comptonization – Why we are interested

From what we introduced last week about the opacities of electron scattering and absorption, we find from the figure below that electron scattering is more important.



Comptonization – Inverse Compton

In each of these photon scatterings, if the electron and photon energies are very different, an inelastic scattering (the Compton process) will occur, either raising or lowering the photon energy in the process. In most cases we will encounter, the photons will be cooler than the electrons, resulting in raising the energy of the photons and, therefore, cooling the hot, electron-scattering plasma in the process.



Comptonization

Since Compton scattering itself doesn’t create new photons, what is does is to use the photons emitted by synchrotron and Bremstrahlung to steal more energy from the system.

Therefore, one way to write the cooling rate is to multiply some Compton-enhancement factor $\eta_C(\nu)$

$$\dot{q} = \eta_{\mathrm{br},C}\dot{q}_{\mathrm{br}}^{-} + \eta_{s,C}\dot{q}_{s}^{-}$$

Comptonization

For the general Compton enhancement factor we use the expression from reference [353]

$$\eta_{\mathrm{C}}(\nu) = e^y\,[1 - \gamma(j_m+1,\ y+s)] + \eta_{\mathrm{C},max}(\nu)\,\gamma(j_m+1,\ s) \quad (9.95)$$

where $\gamma(a,\ x) = [1/\gamma(a)]\int_0^x t^{a-1}\,e^-t\,dt$ is the normalized lower incomplete gamma function (not the Lorentz factor γ) with parameters

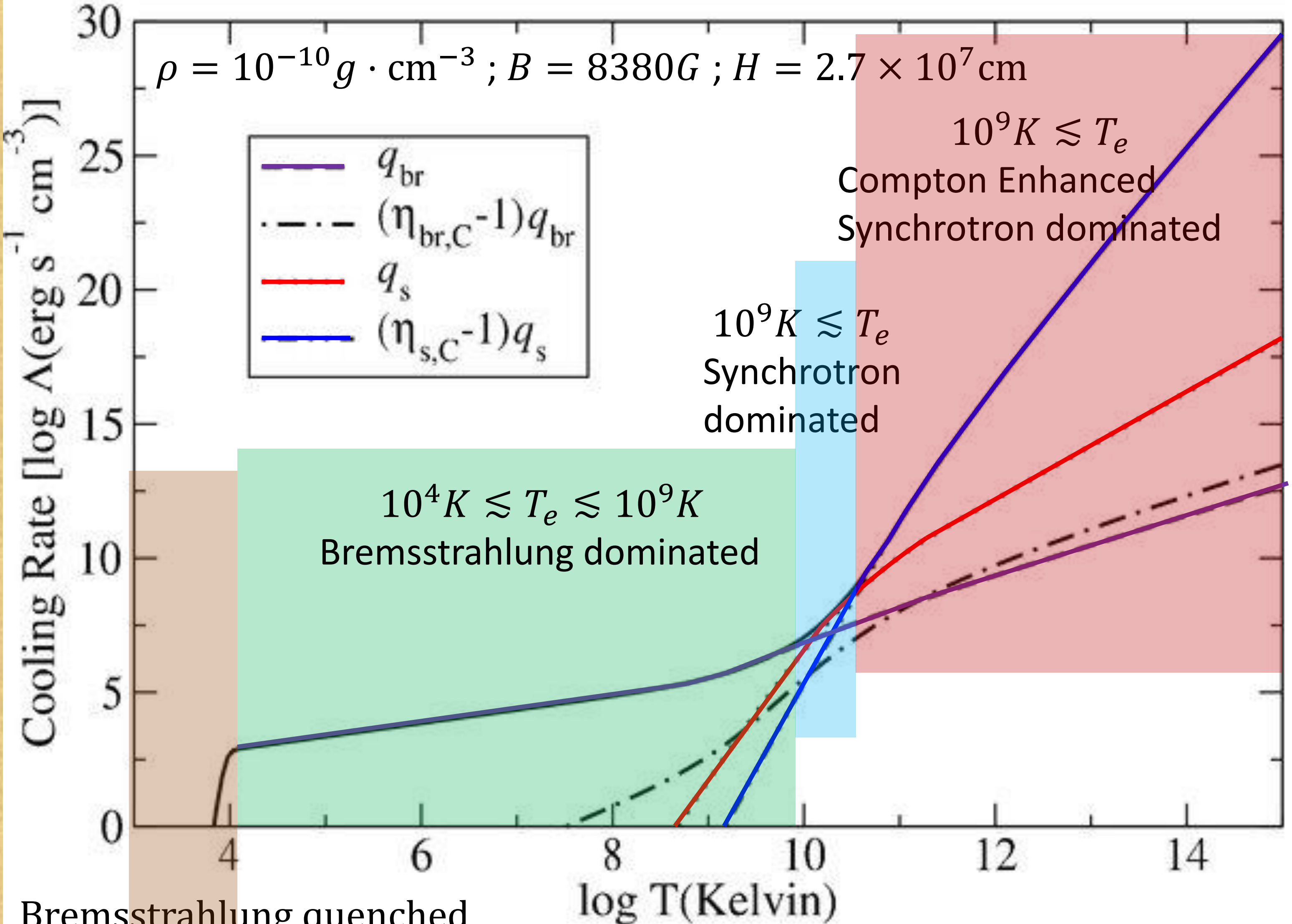
$$A = 1 + 4\,\theta + 16\,\theta^2 \qquad s = \tau_{\mathrm{es}} + \tau_{\mathrm{es}}^2$$

$$j_m = \frac{\ln[\eta_{\mathrm{C},max}(\nu)]}{\ln A} \qquad \eta_{\mathrm{C},max}(\nu) = \frac{3\,kT_{\mathrm{e}}}{h\nu}$$

The famous Compton parameter y is given by

$$y = s\,(A-1) = 4(\theta_{\mathrm{e}} + 4\theta_{\mathrm{e}}^2)\,(\tau_{\mathrm{es}} + \tau_{\mathrm{es}}^2) \quad (9.96)$$

and τ_{es} is the electron scattering optical depth. Note that $\eta_{\mathrm{C}}(\nu)$ is limited to a maximum value of $3kT/h\nu$, where h is Planck's constant. These expressions are most useful in numerical models and simulations.



Bremsstrahlung quenched due to recombination of free electrons with ions.

2009ApJ...693..771F

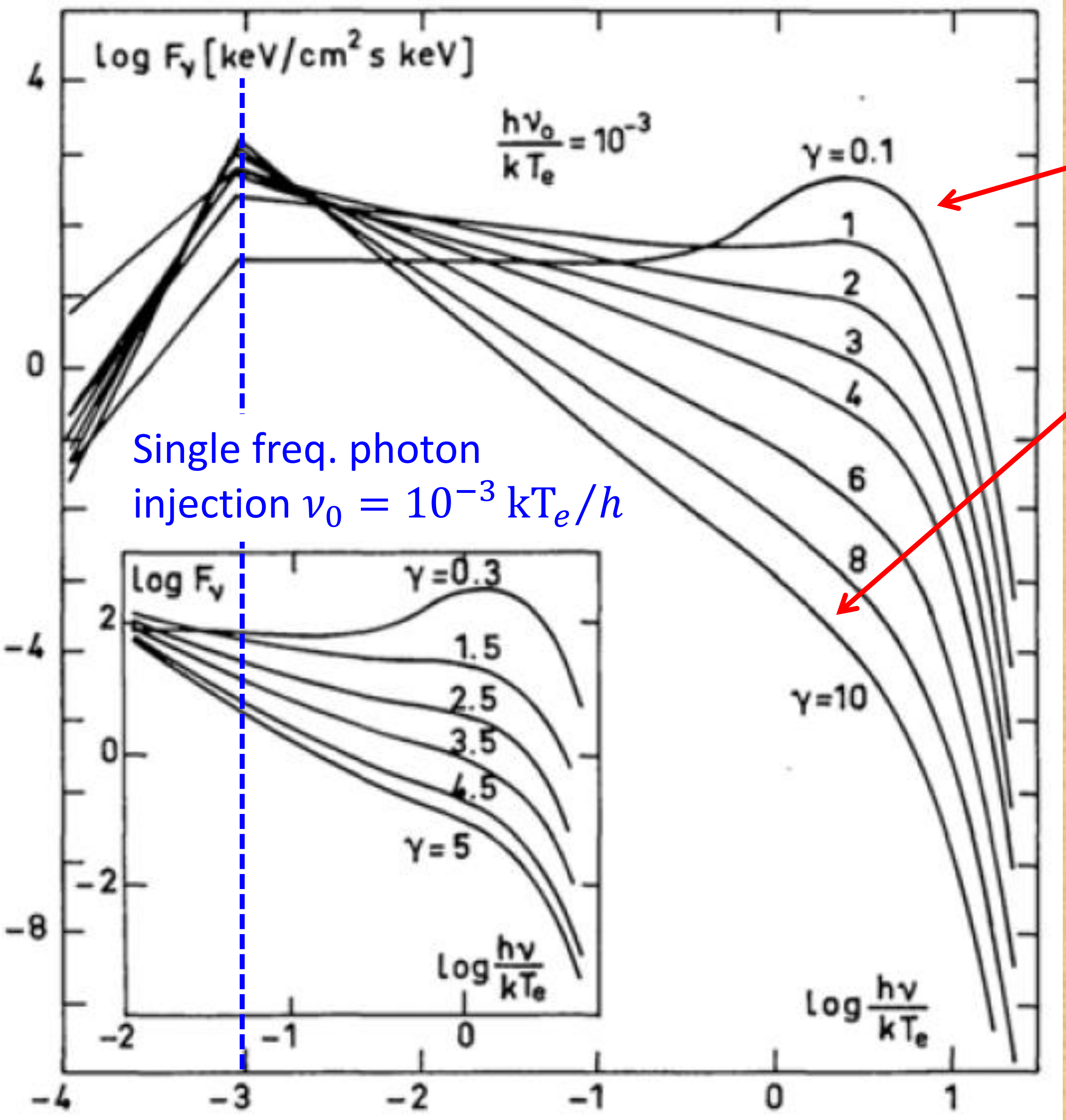
Repeated Scattering

After the initial emission of Bremsstrahlung or synchrotron photons, if the optical depth τ_{es} is high, these photons do not immediately leave the plasma.

They remain within the hot medium, gaining energy from most of the inelastic scatterings they encounter. Depending on how large the Compton y parameter is, this can drastically affect the final spectrum that emerges from the plasma.

$$y \equiv \begin{Bmatrix} \text{average fractional} \\ \text{energy change per} \\ \text{scattering} \end{Bmatrix} \times \begin{pmatrix} \text{mean number of} \\ \text{scatterings} \end{pmatrix}$$

Comptonized spectra for different values of Sunyaev and Titarchuk's parameter $\gamma \approx y^{-1}$



Wien peak is produced as plasma is Compton thick.

Compton thin but still produces a power law spectrum up to the characteristic thermal frequency kT_e/h

As long as the injected photons are much cooler than the hot electron scattering plasma, the nature of the output spectrum from Comptonization depends mainly on y, not on the nature of the input spectrum.